LECTURES ON THE MASSEY VANISHING CONJECTURE

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Abstract. These are lecture notes for the 2024 Workshop on Galois Cohomology and Massey Products, a conference in occasion of Ján Mináč's 71st birthday. We present our joint work on Massey products in Galois cohomology.

Introduction

These are lecture notes from the 2024 Workshop on Galois Cohomology and Massey Products, a conference held in Ottawa on the occasion of Ján Mináč's 71st birthday. At this event, we gave a series of four one-hour lectures on our joint work on Massey products in Galois cohomology [MS22, MS23a, MS25, MS23b]. These notes closely follow the presentation we gave during the conference, which took place June 13-16, 2024.

In Lecture 1, we introduce Massey products in Galois cohomology via Dwyer's theorem, and we state the Massey Vanishing Conjecture (Conjecture 1.6): for every field F, every $n \geq 3$ and every prime p, any non-empty Massey product $\langle \chi_1, \ldots, \chi_n \rangle \subset H^2(F, \mathbb{Z}/p\mathbb{Z})$ of elements $\chi_i \in H^1(F, \mathbb{Z}/p\mathbb{Z})$ contains 0. We recall what is known about this conjecture, and we state our main result (Theorem 1.9): the Massey Vanishing Conjecture holds for fourfold Massey products modulo 2 (this is the subject of [MS23a]). For the proof, it will be convenient to use the language of Galois algebras. We thus conclude Lecture 1 with the interpretation of n-fold Massey products using$n$-fold Massey products using Galois algebras, emphasizing the case of 2-fold Massey productsand we carefully work out the $n = 2$ case (Proposition 1.12).

Lecture 2 begins with a sketch of the proof the Massey Vanishing Conjecture for $n = 3$ using Galois algebras. We then turn to the proof of Theorem 1.9, more precisely, we solve the *degenerate case*, that is, the case where $\chi_1 = \chi_4$ (this was the main goal of [MS22]), and we complete the proof of the general case modulo the key Proposition 2.5.

Lecture 3 is devoted to the proof of Proposition 2.5. The proof given here follows the same general strategy as in [MS23a], but incorporates some simplifications (see the new Proposition 3.2).

In Lecture 4, we discuss the contents of [MS25], where we formalize and investigate the following question: can the known cases of the Massey Vanishing Conjecture be proved using only Hilbert's Theorem 90? We show that this is indeed the case when $n = 3$ (Theorem 4.4) and also when $n = 4$ and $p = 2$ in the degenerate case (Theorem 4.5). We then turn to [MS23b], where we answer a question of Positselski by constructing examples of fields that contain all roots of unity and have non-formal Galois cohomology (Theorem 4.11) with Massey products and

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torsors under tori playing a key role in the proof. Along the way, we also mention our theorem from [MS22] on *doubly degenerate* Massey products (Theorem 4.16), which generalizes an earlier example due to Harpaz–Wittenberg (Example 4.15).

Notation. Let F be a field, let F_s be a separable closure of F, and denote by $\Gamma_F \coloneqq \mathrm{Gal}(F_s/F)$ the absolute Galois group of F. The Γ_F-action on $F_s^\times$ is denoted additively: for all $\sigma, \tau \in \Gamma_F$ and $x \in F_s^\times$, we have $(\sigma+\tau)(x) = \sigma(x)\tau(x)$ and $(\sigma\tau)(x) = \sigma(\tau(x))$.

Let p be a prime, and suppose that $\mathrm{char}(F) \neq p$ and that F contains a primitive p-th root of unity ζ. If E is an étale F-algebra and $a_1, \dots, a_n \in E^\times$, we define the étale E-algebra $E_{a_1,\dots,a_n}$ by

$$E_{a_1,\dots,a_n} \coloneqq E[x_1,\dots,x_n]/(x_1^p - a_1, \dots, x_n^p - a_n)$$

and we set $(a_i)^{1/p} \coloneqq x_i$. More generally, for all integers d, we set $(a_i)^{d/p} \coloneqq x_i^d$.

Let $n \geq 1$. Given $a_1, \dots, a_n \in F^\times$, we fix generators $\sigma_1, \dots, \sigma_n$ of $(\mathbb{Z}/p\mathbb{Z})^n$, and we view the étale F-algebra $F_{a_1,\dots,a_n}$ as a Galois $(\mathbb{Z}/p\mathbb{Z})^n$-algebra, where $\sigma_i(a_i^{1/p}) = \zeta a_i^{1/p}$ and $\sigma_i(a_j^{1/p}) = a_j^{1/p}$ for $i \neq j$. In fact, we will only consider this situation for $n \leq 4$. When $n = 2$, for notational simplicity, we write a, b for a_1, a_2, and σ_a, σ_b for σ_1, σ_2. This is a slight abuse of notation, because $a = b$ does not imply $\sigma_a \neq \sigma_b$: indeed, if $a = b$ then $F_{a,b} = F_a \times F_b$, where σ_a acts trivially on F_b and σ_b acts trivially on F_a. A similar discussion applies to $n = 3$ and $n = 4$.

We write $\mathrm{Br}(F)$ for the Brauer group of F. If F contains a primitive p-th root of unity, for all $a, b \in F^\times$ we denote by (a, b) the corresponding degree-p cyclic algebra over F, as well as its Brauer class in $\mathrm{Br}(F)$. We denote by $N_{a_1,\dots,a_n}$ the norm map from $F_{a_1,\dots,a_n}$ to F, as well as the corestriction $\mathrm{Br}(F_{a_1,\dots,a_n}) \to \mathrm{Br}(F)$.

1. Lecture 1. The Massey Vanishing Conjecture

1.1. **Massey products.** Massey products are higher cohomological operations on the cohomology $H^*(A)$ of a differential graded ring A which generalize the cup product. They were introduced in algebraic topology by Massey [Mas58]: here A is the differential graded ring of singular cochains of a topological space, with coefficients in a ring.

Let Γ be a profinite group, and let p be a prime number. In this lecture series, we are interested in Massey products in the group cohomology of profinite groups: here $A = C^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$ is the differential graded ring of continuous cochains of a profinite group with $\mathbb{Z}/p\mathbb{Z}$ coefficients, so that $H^*(A) = H^*(\Gamma, \mathbb{Z}/p\mathbb{Z})$ is the mod p cohomology ring of Γ. The Massey product of elements of $H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$ admits a simple group-theoretic description, due to Dwyer [Dwy75], which we now recall.

Let $n \geq 2$ be an integer, and let U_{n+1} be the subgroup of upper unitriangular matrices in $\mathrm{GL}_{n+1}(\mathbb{Z}/p\mathbb{Z})$, that is, upper triangular matrices with all diagonal entries equal to 1. This is a p-Sylow subgroup of $\mathrm{GL}_{n+1}(\mathbb{Z}/p\mathbb{Z})$. Its center $Z_{n+1} \simeq \mathbb{Z}/p\mathbb{Z}$ consists of those matrices in U_{n+1} which are zero on every non-diagonal entry except possibly for entry $(1, n+1)$ (the top-right corner). We let $\overline{U}_{n+1} \coloneqq U_{n+1}/Z_{n+1}$ denote the factor group: we can think of elements of $\overline{U}_{n+1}$ as upper unitriangular matrices with the top-right corner removed.

For all i and j such that $1 \leq i < j \leq n+1$, we let $u_{i,j} \colon U_{n+1} \to \mathbb{Z}/p\mathbb{Z}$ be the coordinate function corresponding to entry (i, j). For all $(i, j) \neq (1, n+1)$, the $u_{i,j}$

also define coordinate functions on $\overline{U}_{n+1} \to \mathbb{Z}/p\mathbb{Z}$. The functions $u_{i,j}$ are group homomorphisms if $j = i + 1$, but not in general.

We have a diagram of surjective group homomorphisms

$$U_{n+1} \twoheadrightarrow \overline{U}_{n+1} \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^n, \tag{1.1}$$

where the right map is given $(u_{12}, \dots, u_{n,n+1})$, that is, by forgetting all entries except for the first upper diagonal.

Now let Γ be a profinite group, and consider $\mathbb{Z}/p\mathbb{Z}$ as a discrete Γ-module with trivial action. We have

$$H^1(\Gamma, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Z}/p\mathbb{Z}).$$

Let $\chi_1, \dots, \chi_n \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$ be continuous homomorphisms, and define

$$\chi \coloneqq (\chi_1, \dots, \chi_n) \colon \Gamma \to (\mathbb{Z}/p\mathbb{Z})^n.$$

Consider the diagram

$$\begin{tikzcd}
 & & \Gamma \arrow[d, "\chi"] \\
U_{n+1} \arrow[r, two heads] & \overline{U}_{n+1} \arrow[r, two heads] & (\mathbb{Z}/p\mathbb{Z})^n,
\end{tikzcd}$$

where the bottom row is (1.1).

Let $\overline{\rho} \colon \Gamma \to U_{n+1}$ be a (continuous) lift of χ. Such a lift may not exist, or one may get several liftings. Assuming that $\overline{\rho}$ exists, We want to understand when it may be lifted to a continuous homomorphism $\rho \colon \Gamma \to \overline{U}_{n+1}$. Pictorially, we want to determine whether a dashed arrow ρ in the commutative diagram below exists:

$$\begin{tikzcd}
 & & \Gamma \arrow[d, "\chi"] \arrow[dl, "\overline{\rho}"] \arrow[dll, dashed, "\rho"'] \\
U_{n+1} \arrow[r, two heads] & \overline{U}_{n+1} \arrow[r, two heads] & (\mathbb{Z}/p\mathbb{Z})^n.
\end{tikzcd} \tag{1.2}$$

Concretely, $\overline{\rho}$ may be viewed as a matrix

$$\begin{bmatrix}
1 & \overline{\rho}_{12} & \cdots & \overline{\rho}_{1,n} & \square \\
 & 1 & & & \overline{\rho}_{2,n+1} \\
 & & 1 & & \vdots \\
 & & & 1 & \overline{\rho}_{n,n+1} \\
 & & & & 1
\end{bmatrix}$$

where $\overline{\rho}_{ij} \coloneqq u_{ij} \circ \overline{\rho} \colon \Gamma \to \mathbb{Z}/p\mathbb{Z}$ are cochains (that is, functions). The cochains $\overline{\rho}_{ij}$ are homomorphisms when $j = i + 1$, but not in general. The commutativity of the right triangle in (1.2) is equivalent to

$$\overline{\rho}_{i,i+1} = \chi_i \qquad (1 \leq i \leq n - 1).$$

We now express in matrix notation the condition that $\overline{\rho}$ lifts to a homomorphism $\rho \colon \Gamma \to U_{n+1}$. Let $\eta \colon \Gamma \to \mathbb{Z}/p\mathbb{Z}$ be a cochain (that is, a function), and consider the function $\rho\colon \Gamma \to U_{n+1}$ with matrix representation

$$\begin{bmatrix} 1 & \overline{\rho}_{1,2} & \cdots & \overline{\rho}_{1,n} & \eta \\ & 1 & & & \overline{\rho}_{2,n+1} \\ & & 1 & & \vdots \\ & & & 1 & \overline{\rho}_{n,n+1} \\ & & & & 1 \end{bmatrix}.$$

The function ρ is a group homomorphism if and only if

$$\rho(xy) = \rho(x)\rho(y) \quad \text{for all } x, y \in \Gamma$$

By considering the $(1, n+1)$ entry on both sides, this condition is seen to be equivalent to

$$\eta(xy) = \eta(y) + \sum_{i=2}^{n} \overline{\rho}_{1i}(x)\overline{\rho}_{i,n+1}(y) + \eta(x) \quad \text{for all } x, y \in \Gamma. \tag{1.3}$$

Let us set

$$\Delta(\overline{\rho})\colon \Gamma^2 \to \mathbb{Z}/p\mathbb{Z}, \qquad \Delta(\overline{\rho})(x,y) = \sum_{i=2}^{n} \overline{\rho}_{1i}(x)\overline{\rho}_{i,n+1}(y).$$

Using the fact that $\overline{\rho}$ is a homomorphism, one may check that $\Delta(\overline{\rho})$ is 2-cocycle:

$$\Delta(\overline{\rho}) \in Z^2(\Gamma, \mathbb{Z}/p\mathbb{Z}).$$

Note that $\eta(x) + \eta(y) - \eta(xy) = \partial(\eta)(x,y)$, where ∂ denotes the coboundary in group cohomology. Equation (1.3) may thus be rewritten as

$$\Delta(\overline{\rho})(x,y) = \partial(-\eta)(x,y). \tag{1.4}$$

Thus $\Delta(\overline{\rho})$ represents the obstruction to lifting $\overline{\rho}$ to some ρ:

$$\overline{\rho} \text{ lifts to } \rho \iff [\Delta(\overline{\rho})] = 0 \text{ in } H^2(\Gamma, \mathbb{Z}/p\mathbb{Z}).$$

This motivates the following definition.

Definition 1.1. Let Γ be a profinite group, let p be a prime number, let $n \geq 2$ be an integer, and let $\chi_1, \dots, \chi_n \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$. The *mod p Massey product* of $\chi_1, \dots, \chi_n$ is the subset

$$\langle \chi_1, \dots, \chi_n \rangle \coloneqq \{[\Delta(\overline{\rho})] \mid \overline{\rho}\colon \Gamma \to \overline{U}_{n+1} \text{ lifts } \chi\} \subset H^2(\Gamma, \mathbb{Z}/p\mathbb{Z}).$$

We say that $\langle \chi_1, \dots, \chi_n \rangle$ is *defined* if it is non-empty, that is, if and only if there exists a $\overline{\rho}\colon \Gamma \to \overline{U}_{n+1}$ lifting χ.

We say that $\langle \chi_1, \dots, \chi_n \rangle$ *vanishes* if it contains 0, that is, if and only if there exists a $\rho\colon \Gamma \to U_{n+1}$ lifting χ.

It follows from the definition that, for all $n \geq 2$, if a Massey product $\langle \chi_1, \dots, \chi_n \rangle$ vanishes, then it is defined.

Example 1.2. Suppose that $n = 2$, and let $\chi_1, \chi_2 \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$. Then

$$\overline{\rho} = \begin{bmatrix} 1 & \chi_1 & \square \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\langle \chi_1, \chi_2 \rangle = \{\chi_1 \cup \chi_2\}$. Therefore $\langle \chi_1, \chi_2 \rangle$ is defined, and it vanishes if and only if $\chi_1 \cup \chi_2 = 0$ in $H^2(\Gamma, \mathbb{Z}/p\mathbb{Z})$.

1.2. **The Massey Vanishing Conjecture.**

Proposition 1.3. *Let Γ be a profinite group, let p be a prime number, let $n \geq 3$ be an integer, and let $\chi_1, \dots, \chi_n \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$. We have the following implications:*

$$\langle \chi_1, \dots, \chi_n \rangle \text{ vanishes} \Rightarrow \langle \chi_1, \dots, \chi_n \rangle \text{ is defined} \Rightarrow \chi_i \cup \chi_{i+1} = 0 \ (1 \leq i \leq n).$$

Proof. We have already discussed the first implication. One may prove the second implication as follows. As $n \geq 3$, for all $1 \leq i \leq n-1$ the function

$$\pi_i \colon \overline{U}_{n+1} \to U_3, \qquad A \mapsto \begin{bmatrix} 1 & u_{i,i+1}(A) & u_{i,i+2}(A) \\ 0 & 1 & u_{i+1,i+2}(A) \\ 0 & 0 & 1 \end{bmatrix}$$

is a group homomorphism. If $\overline{\rho}\colon \Gamma \to \overline{U}_{n+1}$ is a lift of $(\chi_1, \dots, \chi_n)\colon \Gamma \to (\mathbb{Z}/p\mathbb{Z})^n$, then $\pi_i \circ \overline{\rho}\colon \Gamma \to U_3$ is a lift of $(\chi_i, \chi_{i+1})\colon \Gamma \to (\mathbb{Z}/p\mathbb{Z})^2$. By Example 1.2, this implies that $\chi_i \cup \chi_{i+1} = 0$. □

Since $\overline{U}_3 = (\mathbb{Z}/p\mathbb{Z})^2$, the second implication of Proposition 1.3 is an equivalence for $n = 3$. Apart from this, the implications of Proposition 1.3 cannot be reversed in general. However, as first observed by Hopkins and Wickelgren [HW15], Massey products exhibit remarkable behavior when $\Gamma = \Gamma_F$ is the absolute Galois group of a field F. For all $i \geq 0$, we write $H^i(F, \mathbb{Z}/p\mathbb{Z})$ for $H^i(\Gamma_F, \mathbb{Z}/p\mathbb{Z})$.

Theorem 1.4 (Hopkins–Wickelgren)**.** *Let F be a number field, and let $\chi_1, \chi_2, \chi_3 \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ be such that the Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle \subset H^2(F, \mathbb{Z}/2\mathbb{Z})$ is defined. Then $\langle \chi_1, \chi_2, \chi_3 \rangle$ vanishes.*

This result was later generalized by Mináč and Tân [MT17a] to arbitrary fields.

Theorem 1.5 (Mináč–Tân)**.** *Let F be an arbitrary field, and let $\chi_1, \chi_2, \chi_3 \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ be such that the Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle \subset H^2(F, \mathbb{Z}/2\mathbb{Z})$ is defined. Then $\langle \chi_1, \chi_2, \chi_3 \rangle$ vanishes.*

Mináč and Tân then made the following conjecture.

Conjecture 1.6 (Massey Vanishing Conjecture (Mináč–Tân))**.** *Let F be a field, let $n \geq 3$ be an integer, let p be a prime number, and let $\chi_1, \dots, \chi_n \in H^1(F, \mathbb{Z}/p\mathbb{Z})$. If $\langle \chi_1, \dots, \chi_n \rangle$ is defined, then it vanishes.*

Remark 1.7 (Motivation for Conjecture 1.6). The main motivation for the Massey Vanishing Conjecture comes from the Profinite Inverse Galois Problem: Which profinite groups are absolute Galois groups of fields?

While an answer to this question is unknown, several necessary conditions have been established. For example, the only finite absolute Galois groups are the trivial group and the cylic group of order two (Artin–Schreier).

A much deeper necessary condition is the following. Assume that F contains a primitive p-th root of unity. The Bloch–Kato Conjecture, proved by Voevodsky and Rost, implies that the cohomology ring $H^*(F, \mathbb{Z}/p\mathbb{Z})$ is quadratic: it admits a presentation with generators in degree 1 (corresponding to elements of $F^\times$) and relations in degree 2 (the Steinberg relations).

Remark 1.8 (Characteristic p). Let p be a prime, and let F be a field such that $H^2(F, \mathbb{Z}/p\mathbb{Z}) = 0$. Then, for all $n \geq 2$, the map $H^1(F, U_{n+1}) \to H^1(F, \overline{U}_{n+1})$ is surjective, and hence the Massey Vanishing Conjecture holds for F.

In particular, the Massey Vanishing Conjecture holds for fields of cohomological p-dimension at most one. For example, it holds for fields of characteristic p, for C_1 fields, and for finite fields.

The table below summarizes the known results on Conjecture 1.6 in chronological order, according to their announcement dates.

F	n	p	Authors	Ref.
Number field	3	2	Hopkins–Wickelgren	[HW15]
Arbitrary	3	2	Mináč–Tân	[MT17a]
Number field	3	Any	Mináč–Tân	[MT15]
Arbitrary	3	Any	Efrat–Matzri, Mináč–Tân	[EM17] [MT16]
Number fields	4	2	Guillot–Mináč–Topaz–Wittenberg	[GMT18]
Number fields	Any	Any	Harpaz–Wittenberg	[HW23]
Arbitrary	4	2	Merkurjev–Scavia	[MS23a]

The goal of Lectures 2 and 3 is to explain the proof of the statement appearing in the bottom row of the table.

Theorem 1.9 (Merkurjev–Scavia)**.** *Let F be a field, let $\chi_1, \chi_2, \chi_3, \chi_4 \in H^1(F, \mathbb{Z}/2\mathbb{Z})$ be such that the Massey product $\langle \chi_1, \chi_2, \chi_3, \chi_4 \rangle \subset H^2(F, \mathbb{Z}/2\mathbb{Z})$ is defined. Then $\langle \chi_1, \chi_2, \chi_3, \chi_4 \rangle$ vanishes.*

1.3. **The case when $\mu_p \subset F^\times$.** Suppose that F contains primitive p-th root of unity $\zeta \in F^\times$. We identify $\mathbb{Z}/p\mathbb{Z}$ and μ_p by means of the isomorphism sending 1 to ζ. Kummer theory gives the identifications

$$H^1(F, \mathbb{Z}/p\mathbb{Z}) = F^\times/F^{\times p}, \qquad H^2(F, \mathbb{Z}/p\mathbb{Z}) = \mathrm{Br}(F)[p].$$

If $a \in F^\times$, we let $\chi_a \colon \Gamma_F \to \mathbb{Z}/p\mathbb{Z}$ be the corresponding continuous homomorphism, that is, letting $a' \in F_s^\times$ be a p-th root of a, the homomorphism χ_a is determined by the equality

$$(g-1)(a') = \zeta^{\chi_a(g)} \quad \text{for all } g \in \Gamma_F.$$

(Recall that we use additive notation for the Galois action on $F_s^\times$.) Under these identifications, the cup product $\chi_a \cup \chi_b$ corresponds to (a, b), the Brauer class of the degree-p cyclic algebra determined by a and b.

We may thus restate Proposition 1.3 for $\Gamma = \Gamma_F$ in the following equivalent form, expressed purely in terms of F.

Proposition 1.10. *Let p be a prime, let F be a field containing a primitive p-th root of unity $\zeta \in F^\times$, let $n \geq 3$ be an integer, and let $a_1, \dots, a_n \in F^\times$. We have the following implications:*

$$\langle a_1, \dots, a_n \rangle \textit{ vanishes} \Rightarrow \langle a_1, \dots, a_n \rangle \textit{ is defined} \Rightarrow (a_i, a_{i+1}) = 0 \ (1 \leq i \leq n).$$

1.4. **Galois algebras.** Let G be a finite group equipped with the trivial Γ_F-action. Recall that the pointed cohomology set $H^1(F, G)$ consists of equivalence classes of continuous homomorphisms $\Gamma_F \to G$, where two homomorphisms $f_1, f_2 \colon \Gamma_F \to G$ are considered equivalent if there exists $g \in G$ such that

$$f_2(\sigma) = g f_1(\sigma) g^{-1} \quad \text{for all } \sigma \in \Gamma_F.$$

The pointed set $H^1(F, G)$ also parametrizes Galois G-algebras. By definition, a *G-algebra* L/F is an étale F-algebra equipped with a G-action by F-algebra automorphisms. The G-algebra L is said to be *Galois* if $|G| = \dim_F L$ and $L^G = F$; see [KMRT98, Definitions (18.15)]. A G-algebra L/F is Galois if and only if the corresponding morphism of schemes $\mathrm{Spec}(L) \to \mathrm{Spec}(F)$ is an étale G-torsor. The automorphism group of a trivial G-algebra over F (equivalently, of a trivial G-torsor) may be identified to G, and hence by Galois descent we have a canonical bijection

$$H^1(F, G) \xrightarrow{\sim} \{\text{Isomorphism classes of Galois } G\text{-algebras over } F\} \tag{1.5}$$

which is functorial in F and G; see [KMRT98, Example (28.15)].

Conjecture 1.6 may be restated as saying that every Galois $(\mathbb{Z}/p\mathbb{Z})^n$-algebra which is induced by a $\overline{U}_{n+1}$-algebra is also induced by a U_{n+1}-algebra. More precisely, let Q_{n+1} be the kernel of the projection $U_{n+1} \to (\mathbb{Z}/p\mathbb{Z})^n$, and let $\overline{Q}_{n+1} \coloneqq Q_{n+1}/Z_{n+1}$.

Conjecture 1.11 (Massey Vanishing Conjecture, Galois algebra formulation). *Let F be a field, let p be a prime, let $n \geq 3$, and let K/F be a Galois $(\mathbb{Z}/p\mathbb{Z})^n$-algebra. If there exists a Galois $\overline{U}_{n+1}$-algebra E/F such that $E^{\overline{Q}_{n+1}} = K$ as $(\mathbb{Z}/p\mathbb{Z})^n$-algebras, then there exists a Galois U_{n+1}-algebra L/F such that $L^{Q_{n+1}} = K$ as $(\mathbb{Z}/p\mathbb{Z})^n$-algebras.*

Suppose now that $G = N \rtimes S$, where N is a normal subgroup of G and S is a subgroup of G. Let E be a Galois S-algebra over F, and let $\pi\colon \Gamma_F \to S$ be a continuous group homomorphism whose class in $H^1(F, S)$ coincides with the class of E/F. The group S acts on N by conjugation. We view N as a Γ_F-module via π, and we write N_π for the twist; see [KMRT98, 28.C]. We have a canonical bijection

$$\begin{aligned} H^1(F, N_\pi) \xrightarrow{\sim} \{&\text{Isom. classes of pairs } (K, \varphi), \text{ where } K/F \text{ is a Galois } G\text{-algebra} \\ &\text{and } \varphi\colon K^N \to E \text{ is an isomorphism of Galois } S\text{-algebras}\}, \end{aligned} \tag{1.6}$$

which is functorial in F. Here, an isomorphism of pairs $(K, \varphi) \to (K', \varphi')$ is defined as an isomorphism of Galois G-algebras $\sigma\colon K \to K'$ over F such that $\varphi = \varphi' \circ \sigma$ on K^N. We recall how the bijection (1.6) is constructed. Letting K_0 be the Galois G-algebra induced by E, we have a canonical isomorphism $\varphi_0\colon K_0^N \xrightarrow{\sim} E$ of Galois S-algebras, and the automorphism group of the pair (K_0, φ_0) is naturally identified with N_π: by Galois descent, this defines (1.6). Under the identification (1.6), the surjective twisting map

$$H^1(F, N_\pi) \to \mathrm{Fiber}_\pi[H^1(F, G) \to H^1(F, S)] \tag{1.7}$$

of [KMRT98, Proposition 28.11] sends the class of a pair (K, φ) to the class of K.

1.5. **Galois U_3-algebras.** For the remainder of this lecture, we assume that $p = 2$. In particular, the groups U_3, Z_3 and $\overline{U}_3$ introduced earlier will now be considered with respect to the prime $p = 2$. We give an equivalent interpretation of Example 1.2 (for $p = 2$) in terms of Galois U_3-algebras, which will be used in the proof of Proposition 2.4, and hence in the proof of Theorem 1.9.

Suppose that $\mathrm{char}(F) \neq 2$, let $a, b \in F^\times$, and suppose that $(a, b) = 0$ in $\mathrm{Br}(F)[2]$. We write $(\mathbb{Z}/2\mathbb{Z})^2 = \langle \sigma_a, \sigma_b \rangle$, and we view $F_{a,b}$ as a Galois $(\mathbb{Z}/2\mathbb{Z})^2$-algebra over F via the action

$$(\sigma_a - 1)(\sqrt{a}) = (\sigma_b - 1)(\sqrt{b}) = -1, \qquad (\sigma_a - 1)(\sqrt{b}) = (\sigma_b - 1)(\sqrt{a}) = 1.$$

Let $\alpha \in F_a^\times$ satisfy $N_a(\alpha) = bx^2$ for some $x \in F^\times$, and consider the étale F-algebra $(F_{a,b})_\alpha$. We have

$$U_3 = \langle \sigma_a, \sigma_b : \sigma_a^2 = \sigma_b^2 = [\sigma_a, \sigma_b]^2 = 1 \rangle,$$

where the commutator $[\sigma_a, \sigma_b]$ generates the center $Z_3 \subset U_3$. It follows that we may identify $\overline{U}_3 = (\mathbb{Z}/2\mathbb{Z})^2$ in such a way that the surjective homomorphism $U_3 \to \overline{U}_3$ is given by $\sigma_a \mapsto \sigma_a$ and $\sigma_b \mapsto \sigma_b$. Observe that $\sigma_a(\alpha) = bx^2/\alpha$ and $\sigma_b(\alpha) = \alpha$. We may thus define a Galois U_3-algebra structure on $(F_{a,b})_\alpha$ by letting U_3 act on $F_{a,b}$ via $\overline{U}_3$ and by setting

$$\sigma_a(\sqrt{\alpha}) = x\sqrt{b}/\sqrt{\alpha}, \qquad \sigma_b(\sqrt{\alpha}) = \sqrt{\alpha}. \tag{1.8}$$

One verifies that $\sigma_a^2 = \sigma_b^2 = [\sigma_a, \sigma_b]^2 = 1$ on $(F_{a,b})_\alpha$, that $(F_{a,b})_\alpha$ is a Galois U_3-algebra and that its subalgebra of Z_3-invariants is $F_{a,b}$.

Symmetrically, if $\beta \in F_b^\times$ satisfies $N_b(\beta) = ay^2$ for some $y \in F^\times$, the étale F-algebra $(F_{a,b})_\beta$ has the structure of a Galois U_3-algebra defined by

$$\sigma_a(\sqrt{\beta}) = \sqrt{\beta}, \qquad \sigma_b(\sqrt{\beta}) = y\sqrt{a}/\sqrt{\beta}. \tag{1.9}$$

Proposition 1.12. *Assume that $p = 2$, let F be a field of characteristic different from 2, and let $a, b \in F^\times$ be such that $(a, b) = 0$ in $\mathrm{Br}(F)[2]$.*

(a) Every Galois U_3-algebra K over F such that $K^{Z_3} = F_{a,b}$ is of the form $(F_{a,b})_\alpha$ for some $\alpha \in F_a^\times$ with the property $N_a(\alpha) = b$ in $F^\times/F^{\times 2}$ and U_3-algebra structure as in (1.8).

(b) Every Galois U_3-algebra K over F such that $K^{Z_3} = F_{a,b}$ is of the form $(F_{a,b})_\beta$ for some $\beta \in F_b^\times$ with the property $N_b(\beta) = a$ in $F^\times/F^{\times 2}$ and U_3-algebra structure as in (1.9).

Proof. (a) We have $U_3 = N \rtimes S$, where

$$N = \begin{bmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & * & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}.$$

We let S act on N by conjugation. As an S-module, N has a permutation basis given by

$$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}.$$

We obtain a commutative square of S-modules:

$$\begin{tikzcd} N \arrow[r, "\sim"] \arrow[d, "u_{23}"] & \mathrm{Ind}_{\{1\}}^{S}(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\mathrm{Norm}"] \\ \mathbb{Z}/2\mathbb{Z} \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z}, \end{tikzcd} \tag{1.10}$$

where Ind denotes the induction functor. Let $\mathrm{pr}\colon U_3 \to S$ be the projection map, and let N_{pr} be the twist of N by pr; see [KMRT98, 28.C]. Then (1.10) induces a commutative square of U_3-modules:

$$\begin{tikzcd} N_{\mathrm{pr}} \arrow[r, "\sim"] \arrow[d, "u_{23}"] & \mathrm{Ind}_{N}^{U_3}(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\mathrm{Norm}"] \\ \mathbb{Z}/2\mathbb{Z} \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z}. \end{tikzcd} \tag{1.11}$$

Let $\rho\colon \Gamma_F \to U_3$ be a continuous homomorphism whose class in $H^1(F, U_3)$ coincides with that of K, and define $\pi\colon \Gamma_F \to S$ by $\pi \coloneqq p \circ \rho$. We have $K^{Z_3} = F_{a,b}$, and hence $K^N = (K^{Z_3})^{\sigma_b} = (F_{a,b})^{\sigma_b} = F_a$. In turn, the equality $K^N = F_a$ implies that the class of π in $H^1(F, S) = H^1(F, \mathbb{Z}/2\mathbb{Z})$ is equal to χ_a. Pullback of (1.11) along ρ yields the following commutative square:

$$\begin{tikzcd} N_\pi \arrow[r, "\sim"] \arrow[d, "u_{23}"] & \mathrm{Ind}^F_{F_a}(\mathbb{Z}/2\mathbb{Z}) \arrow[d, "N_a"] \\ \mathbb{Z}/2\mathbb{Z} \arrow[r, equal] & \mathbb{Z}/2\mathbb{Z}. \end{tikzcd} \tag{1.12}$$

Here $\mathrm{Ind}^F_{F_a}(\mathbb{Z}/2\mathbb{Z})$ indicates the Γ_F-module corresponding to the pushforward of the constant étale sheaf $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{Spec}(F_a)$ to $\mathrm{Spec}(F)$; see [Mil80, Theorem II.1.9]. (Concretely, when F_a is a field we have $\mathrm{Ind}^F_{F_a}(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Ind}^{\Gamma_F}_{\Gamma_{F_a}}(\mathbb{Z}/2\mathbb{Z})$, and when $F_a = F \times F$ we have $\mathrm{Ind}^F_{F_a}(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with trivial action.) Combining (1.12) with Faddeev–Shapiro's lemma [NSW08, Proposition 1.6.4], we deduce that the composition

$$\Phi\colon H^1(F, N_\pi) \xrightarrow{\mathrm{Res}} H^1(F_a, N) \xrightarrow{u_{13}} H^1(F_a, \mathbb{Z}/2\mathbb{Z})$$

is an isomorphism fitting in the commutative diagram

$$\begin{tikzcd} H^1(F, N_\pi) \arrow[r, "\Phi"] \arrow[d, "u_{23}"] & H^1(F_a, \mathbb{Z}/2\mathbb{Z}) \arrow[r, equal] \arrow[d, "N_a"] & F_a^\times / F_a^{\times 2} \arrow[d, "N_a"] \\ H^1(F, \mathbb{Z}/2\mathbb{Z}) \arrow[r, equal] & H^1(F, \mathbb{Z}/2\mathbb{Z}) \arrow[r, equal] & F^\times / F^{\times 2}. \end{tikzcd} \tag{1.13}$$

Let E be a Galois U_3-algebra and $\varphi\colon E^N \to F_a$ be an isomorphism of Galois U_3-algebras over F. By base change, we obtain an isomorphism of Galois U_3-algebras

$$\varphi_{F_a}\colon (E^N)_a \xrightarrow{\sim} (F_a)_a = \prod_{\sigma \in S} F_a$$

over F_a. Therefore, we may write $E_a = \prod_{\sigma \in S} E_{\varphi,\sigma}$, where $E_{\varphi,\sigma}$ is the subalgebra of E_a lying over the inverse image of the factor in $\prod_{\sigma \in S} F_a$ corresponding to σ. In terms of the identification (1.6), the Faddeev–Shapiro isomorphism Φ sends the class of the pair (E, φ) to the class of the Galois $\mathbb{Z}/2\mathbb{Z}$-algebra $E_{\varphi,0}/F_a$, where $0 \in S$ is the identity element.

Since $K^{Z_3} = F_{a,b}$, the pair (K, id) defines an element in $H^1(F, N_\pi)$. Let $\alpha \in F_a^\times$ be such that Φ sends the class of (K, id) to $(F_a)_\alpha/F_a$. By (1.13), we have $N_a(\alpha) = bx^2$ for some $x \in F^\times$. The pair $((F_{a,b})_\alpha, \mathrm{id})$, where $(F_{a,b})_\alpha$ is the Galois U_3-algebra of (1.8), also defines an element of $H^1(F, N_\pi)$ which is mapped to $(F_a)_\alpha/F_a$ by Φ. The injectivity of Φ now implies that the Galois U_3-algebras K and $(F_{a,b})_\alpha$ are isomorphic over F, as desired.

(b) Analogous to (a), replacing N and S by

$$N' = \begin{bmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{bmatrix} \quad \text{and} \quad S' = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & * \\ & & 1 \end{bmatrix},$$

respectively. □

2. Lecture 2. Beginning of Proof of Theorem 1.9

2.1. **Proof of Massey vanishing for $n = 3$.** As as warm-up for the proof of Theorem 1.9, we sketch a new proof of the case $n = 3$ of the Massey Vanishing Conjecture. (In the $p = 2$ case, we will give a complete proof.) See [MS22, Corollary 3.4] for an alternative proof using Galois U_4-algebras, and [MS25, Theorem 1.3] for an alternative proof using only cocycles.

Theorem 2.1 (Efrat–Matrzi, Mináč–Tân)**.** *Let F be a field, let p be a prime number, and let $\chi_1, \chi_2, \chi_3 \in H^1(F, \mathbb{Z}/p\mathbb{Z})$. The following are equivalent:*

(1) $\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = 0$ in $H^2(F, \mathbb{Z}/p\mathbb{Z})$;
(2) $\langle \chi_1, \chi_2, \chi_3 \rangle$ is defined;
(3) $\langle \chi_1, \chi_2, \chi_3 \rangle$ vanishes.

Proof. We know that (3) implies (2), and that (1) and (2) are equivalent (see below Proposition 1.3). It remains to prove that (2) implies (3). By Remark 1.8, we may assume that $\operatorname{char}(F) \neq p$. By a simple argument [MT16, Proposition 4.14] which uses the fact that the degree $[F(\mu_p) : F]$ is prime to p, we may also assume that F contains a primitive p-th root of unity ζ.

Let $a, b, c \in F^\times$ be such that $\chi_1 = \chi_a$, $\chi_2 = \chi_b$ and $\chi_3 = \chi_c$. Since $\langle a, b, c \rangle$ is defined, we have a homomorphism $\overline{\rho}\colon \Gamma_F \to \overline{U}_4$ of the form

$$\begin{bmatrix} 1 & \chi_a & \mu & \square \\ 0 & 1 & \chi_b & \theta \\ 0 & 0 & 1 & \chi_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The obstruction to lifting this homomorphism to U_4 is given by

$$A \coloneqq \Delta(\overline{\rho}) = \chi_a \cup \theta + \mu \cup \chi_c \in \operatorname{Br}(F)[p].$$

Consider the following assumption.

Assumption 2.2. The Brauer class A is decomposable: there exist $x, y \in F^\times$ such that

$$A = (a, x) + (y, c) = \chi_a \cup \chi_x + \chi_y \cup \chi_c \in \operatorname{Br}(F)[p].$$

We prove (3) under Assumption 2.2. Consider the continuous homomorphism $\overline{\rho}'\colon \Gamma_F \to \overline{U}_4$ defined by the matrix

$$\begin{bmatrix} 1 & \chi_a & \mu - \chi_y & \square \\ 0 & 1 & \chi_b & \theta - \chi_x \\ 0 & 0 & 1 & \chi_c \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The obstruction to lifting this homomorphism to U_4 is given by

$$\chi_a \cup (\theta - \chi_x) + (\mu - \chi_y) \cup \chi_c = A - A = 0,$$

and hence $\overline{\rho}'$ lifts to U_4. Thus $\langle a, b, c \rangle$ vanishes, as desired.

In order to complete the proof, it remains to verify Assumption 2.2. Let

$$\operatorname{Br}(F_{a,c}/F) \coloneqq \operatorname{Ker}[\operatorname{Br}(F) \to \operatorname{Br}(F_{a,c})]$$

and let $\operatorname{Dec} \subset \operatorname{Br}(F_{a,c}/F)$ be the subgroup of decomposable elements, that is, elements of the form $(a, x) + (y, c)$ for some $x, y \in F^\times$. The cocycle description of A implies that $A \in \operatorname{Br}(F_{a,c}/F)$, and we want to show that $A \in \operatorname{Dec}$. If $p = 2$, then

$\mathrm{Dec} = \mathrm{Br}(F_{a,c}/F)[2]$, and hence A is decomposable. (This is property $P_2(2)$ for F in the sense of [ELTW83, p. 1129], and it is satisfied by [ELTW83, Remark 3.12].) When $p > 2$, it is no longer true that $\mathrm{Dec} = \mathrm{Br}(F_{a,c}/F)[p]$ in general. However, letting $G_{a,c} := \mathrm{Gal}(F_{a,c}/F)$, we have a canonical isomorphism

$$\mathrm{Br}(F_{a,c}/F)[p]/\mathrm{Dec} \xrightarrow{\sim} \hat{H}^{-1}(G_{a,c}, F_{a,c}^{\times}) = \frac{\{z \in F_{a,c}^{\times} : N_{a,c}(z) = 1\}}{\{(\sigma - 1)u : \sigma \in G_{a,c}, u \in F_{a,c}^{\times}\}}.$$

Since $(a, b) = (b, c) = 0$, there exist $\alpha \in F_a^{\times}$ and $\gamma \in F_c^{\times}$ such that $N_a(\alpha) = b$ and $N_c(\gamma) = b$. One can show by direct calculation that the image of $\Delta(\overline{\rho})$ is the class of $z = \gamma/\alpha$ in $\hat{H}^1$. We have $N_{F_{a,c}/F_{ac}}(z) = N_a(\alpha)/N_c(\gamma) = b/b = 1$. By Hilbert's Theorem 90, this implies that $z = (\sigma - 1)(u)$ for some $u \in F_{a,c}^{\times}$, where σ is a generator of $\mathrm{Gal}(F_{a,c}/F_{ac})$. Thus the class of z in $\hat{H}^1$ is trivial. Therefore, $A = \Delta(\overline{\rho})$ is decomposable and Assumption 2.2 holds, as desired. □

2.2. **Beginning of proof of Theorem 1.9.** We begin the proof of Theorem 1.9. From now on, in this lecture and the next, we assume that $p = 2$. In particular, the groups U_n, Z_n and $\overline{U}_n$ introduced in Lecture 1 will be considered with respect to the prime $p = 2$.

In view of Remark 1.8, we may suppose that $\mathrm{char}(F) \neq 2$, so that Theorem 1.9 can be restated as follows:

Theorem 2.3. *Let F be a field of characteristic not 2, let $a, b, c, d \in F^{\times}$ be such that the mod 2 Massey product $\langle a, b, c, d\rangle$ is defined. Then $\langle a, b, c, d\rangle$ vanishes.*

The starting point for the proof is the following proposition, which characterizes the properties "$\langle a, b, c, d\rangle$ vanishes" and "$\langle a, b, c, d\rangle$ is defined" using the Brauer group of $F_{a,d}$. Part (1) can essentially be found in [GMT18].

Proposition 2.4 (Guillot–Mináč–Topaz–Wittenberg)**.** *Let $a, b, c, d \in F^{\times}$.*

1. *The Massey product $\langle a, b, c, d\rangle$ vanishes if and only if there exist $\alpha \in F_a^{\times}$ and $\delta \in F_d^{\times}$ such that $N_a(\alpha) = b$ and $N_d(\delta) = c$ in $F^{\times}/F^{\times 2}$, and $(\alpha, \delta) = 0$ in $\mathrm{Br}(F_{a,d})[2]$.*
2. *The Massey product $\langle a, b, c, d\rangle$ is defined if and only if there exist $\alpha \in F_a^{\times}$ and $\delta \in F_d^{\times}$ such that $N_a(\alpha) = b$ and $N_d(\delta) = c$ in $F^{\times}/F^{\times 2}$, and (α, δ) belongs to the image of the restriction map $\mathrm{Br}(F)[2] \to \mathrm{Br}(F_{a,d})[2]$.*

Given $\alpha \in F_a^{\times}$, $\delta \in F_d^{\times}$ such that $N_a(\alpha) = b$, $N_d(\delta) = c$ in $F^{\times}/F^{\times 2}$ and $(\alpha, \delta) \in \mathrm{Im}(\mathrm{Br}(F)[2] \to \mathrm{Br}(F_{a,d})[2])$, we will replace α by αx and δ by δy, for suitable $x, y \in F^{\times}$, such that $(\alpha x, \delta y) = 0$ in $\mathrm{Br}(F_{a,d})[2]$. We will accomplish this in two steps, which as a first approximation may be summarized as:

1. Reduce to the degenerate case $a = d$, by replacing $\alpha \mapsto \alpha x$ and $\delta \in F_d^{\times}$ by an appropriate $\nu \in F_a^{\times}$.
2. Solve the degenerate case $a = d$, by replacing δ by δy, for some $y \in F^{\times}$.

Here are the precise versions of the two steps.

Proposition 2.5. *Let $a, c, d \in F^{\times}$, let $\alpha \in F_a^{\times}$, let $\delta \in F_d^{\times}$ such that $N_d(\delta) = c$ in $F^{\times}$ and (α, δ) is in the image of* $\mathrm{Br}(F)[2] \to \mathrm{Br}(F_{a,d})[2]$. *Suppose that c is not a square in $F^{\times}$. Then there exist $x \in F^{\times}$ and $\nu \in F_a^{\times}$ such that $(\alpha x, \delta) = (\alpha x, \nu)$ in* $\mathrm{Br}(F_{a,d})[2]$ *and $(\alpha x, \nu)$ is in the image of the restriction map* $\mathrm{Br}(F)[2] \to \mathrm{Br}(F_a)[2]$.

Proposition 2.6. *Let $a \in F^{\times}$, let $\pi, \mu \in F_a^{\times}$ such that $N_a(\pi, \mu) = 0$ in* $\mathrm{Br}(F)[2]$. *Then there exists $y \in F^{\times}$ such that $(\pi, \mu y) = 0$ in* $\mathrm{Br}(F_a)$.

Proof of 2.4+2.5+2.6 ⇒ Theorem 1.9. It suffices to prove Theorem 2.3. The case when c is a square in F is not difficult and is left to the reader. From now on, we suppose that c is not a square in F. Since $\langle a,b,c,d\rangle$ is defined, Proposition 2.4(1) gives $\alpha \in F_a^\times$, $\delta \in F_d^\times$ such that $N_a(\alpha) = b$ and $N_d(\delta) = c$ in $F^\times/F^{\times 2}$ and $(\alpha,\delta) \in \operatorname{Im}(\operatorname{Br}(F)[2] \to \operatorname{Br}(F_{a,d})[2])$. Proposition 2.5 shows that

$$(\alpha x,\delta) = (\alpha x,\nu) \quad \text{in } \operatorname{Br}(F_{a,d})[2], \qquad N_a(\alpha x,\nu) = 0 \quad \text{in } \operatorname{Br}(F)[2]$$

for some $x \in F^\times$ and $\nu \in F_a^\times$. If we set $\pi = \alpha x$ and $\mu = \nu$, then $N_a(\pi,\mu) = 0$. Proposition 2.6 then gives $y \in F^\times$ such that

$$(\alpha x,\nu y) = 0 \qquad \text{in } \operatorname{Br}(F_a)[2].$$

Putting it all together, we get the following string of equalities in $\operatorname{Br}(F_{a,d})[2]$:

$$(\alpha x,\delta y) = (\alpha x,\delta) + (\alpha x,y) = (\alpha x,\nu) + (\alpha x,y) = (\alpha x,\nu y) = 0.$$

By Proposition 2.4(2), the Massey product $\langle a,b,c,d\rangle$ vanishes. ☐

In this lecture, we prove Propositions 2.4 and 2.6. In the next lecture, we will prove Proposition 2.5, and hence complete the proof of Theorem 1.9.

Proof sketch of Proposition 2.4. We sketch the proof given in [MS23a, Proposition 2.1]. The proof is based on the following commutative diagram of groups with exact rows and columns

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} \arrow[r, equal] \arrow[d, hook] & \mathbb{Z}/2\mathbb{Z} \arrow[d, hook] & \\
P \arrow[r, hook] \arrow[d, two heads] & U_5 \arrow[r, two heads] \arrow[d, two heads] & U_3 \times U_3 \arrow[d, equal] \\
\overline{P} \arrow[r, hook] & \overline{U}_5 \arrow[r, two heads] & U_3 \times U_3.
\end{tikzcd} \tag{2.1}$$

Here P is the normal subgroup of U_5 given by

$$P := \begin{bmatrix} 1 & 0 & 0 & * & * \\ & 1 & 0 & * & * \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}.$$

Define $\chi := (\chi_a,\chi_b,\chi_c,\chi_d)\colon \Gamma_F \to (\mathbb{Z}/2\mathbb{Z})^4$. By Proposition 1.12, the Galois $(U_3 \times U_3)$-algebras K/F such that $K^{Z_3 \times Z_3} = F_{a,b,c,d}$ are all of the form $(F_{a,b})_\alpha \otimes_F (F_{c,d})_\delta$ for some $\alpha \in F_a^\times$ and $\delta \in F_d^\times$ such that $N_a(\alpha) = b$ and $N_d(\delta) = c$ in $F^\times/F^{\times 2}$. We write $(\rho_\alpha,\rho_\delta)\colon \Gamma_F \to U_3 \times U_3$ for the lift of χ corresponding to $(F_{a,b})_\alpha \otimes_F (F_{c,d})_\delta$.

Let N and S be the subgroups of U_3 as in the proof of Proposition 1.12(a). In particular, N is an S-module (by conjugation). Let N' and S' be the corresponding subgroups of U_3 as in the proof of Proposition 1.12(b). The bilinear map

$$N \times N' \to P$$

taking a pair of matrices

$$\begin{bmatrix} 1 & 0 & f_1 \\ & 1 & e_1 \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & e_2 & f_2 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & 0 & 0 & f_1e_2 & f_1f_2 \\ & 1 & 0 & e_1e_2 & e_1f_2 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix}$$

yields an isomorphism of $(U_3 \times U_3)$-modules

$$N \otimes N' \xrightarrow{\sim} P. \tag{2.2}$$

Let Γ_F act on P via $(\rho_\alpha, \rho_\delta)$ and the conjugation $U_3 \times U_3$-action on P. Then (2.2) yields is an isomorphism of Γ_F-modules

$$\mathrm{Ind}_{F_{a,d}}^{F}(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} P.$$

In particular, $H^2(F, P) = H^2(F_{a,d}, \mathbb{Z}/2\mathbb{Z}) = \mathrm{Br}(F_{a,d})[2]$. (Here it is crucial that $p = 2$.) One checks that the obstruction to lifting $(\rho_\alpha, \rho_\delta)$ to U_5 is equal to the Brauer class $(\alpha, \delta) \in \mathrm{Br}(F_{a,d})[2]$. □

Proof of Proposition 2.6. Let A be a biquaternion algebra, that is, A is the tensor product of two quaternion algebras (a_1, b_1) and (a_2, b_2), where $a_1, b_1, a_2, b_2 \in F^\times$. The Albert form of A is the quadratic form $q := \langle a_1, b_1, -a_1b_1, -a_2, -b_2, a_2b_2\rangle$. (This is a quadratic form, not a Massey product!) The Albert form of A depends on the presentation of A as $(a_1, b_1) \otimes (a_2, b_2)$, but its similarity class is uniquely determined.

Let $w(q)$ be the Witt index of q, that is, the dimension of a maximal totally isotropic subspace of q. By a theorem of Albert [KMRT98, Theorem 16.5], A is split if and only if q is hyperbolic. Let:

- $s\colon F_a \to F$ be a non-zero F-linear map such that $s(1) = 0$;
- Q be the quaternion algebra (π, μ);
- $Q^\circ \subset Q$ the subspace of pure quaternions;
- $q\colon Q^\circ \to F_a$ be the quadratic form given by $q(x) = x^2$. A computation shows that $q = \langle \pi, \mu, -\pi\mu\rangle$;
- $s_*(q)\colon Q^\circ \to F_a \xrightarrow{s} F$ the Scharlau transfer of q.

By another theorem of Albert, $s_*(q)$ is similar to an Albert form for $N_a(Q)$; see the proof of [KMRT98, Corollary (16.28)]. By assumption, $N_a(Q)$ is split, and hence $s_*(q)$ is hyperbolic. Thus $s_*\langle \mu, -\pi\mu\rangle$ is a 4-dimensional subform of a 6-dimensional hyperbolic form. Since $4 > 6/2$, this implies that $s_*\langle \mu, -\pi\mu\rangle$ is isotropic:

$$\text{There exist } p, q \in F_a^\times, \text{ and } z \in F \text{ such that } \mu p^2 - \pi\mu q^2 = z.$$

If $z = 0$, then π is a square and we may take $y = 1$. If $z \neq 0$ then, multiplying the previous equation by μ gives $(\mu p)^2 - \pi(\mu q)^2 = \mu z$, so that $\mu z \in F_a^\times$ is a norm from $((F_a)_\pi)^\times$. This is equivalent to $(\pi, \mu z) = 0$ in $\mathrm{Br}(F_a)$. Thus we may take $y = z$. □

Remark 2.7. The combination of Proposition 2.4 and Proposition 2.6 implies the Massey Vanishing Conjecture for degenerate fourfold Massey products, that is, Massey products of the form $\langle a, b, c, a\rangle$; see [MS22, Theorem 1.3].

3. Lecture 3. End of proof of Theorem 1.9

3.1. **Specialization in Galois cohomology.** Recall from [Ros96, Remarks 1.11 and 2.5] that the Galois cohomology functor $H^*(-,\mathbb{Z}/2\mathbb{Z})$ from the category of field extensions of F is a cycle module, that is, it satisfies the axioms of [Ros96, Definitions 1.1 and 2.1].

For all integers $n \geq 1$, all regular local F-algebras R of dimension n and all regular systems of parameters $\pi \coloneqq (\pi_1,\dots,\pi_n)$ in R, letting K and $K_0 \coloneqq R/(\pi_1,\dots,\pi_n)$ be the fraction field and residue field of R, respectively, we have a graded ring homomorphism

$$s_\pi\colon H^*(K,\mathbb{Z}/2\mathbb{Z}) \to H^*(K_0,\mathbb{Z}/2\mathbb{Z}),$$

called the specialization map, which is defined as follows.

Suppose first that $n=1$, that is, R is a discrete valuation ring and $\pi=(\pi_1)$. Then we set $s_\pi \coloneqq \partial_{\pi_1}((-\pi_1)\cup(-))$, where $\partial_{\pi_1}\colon H^{*+1}(K,\mathbb{Z}/2\mathbb{Z}) \to H^*(K_0,\mathbb{Z}/2\mathbb{Z})$ is the residue map at π_1; see [Ros96, Definition 1.1, below D4].

Suppose now that $n \geq 2$ and that the specialization map has been defined for all regular local F-algebras of dimension $< n$ and all regular systems of parameters on such algebras. For $i = 2,\dots,n$ let $\overline{\pi}_i \in R/(\pi_1)$ be the reduction of π_i modulo π_1 and set $\overline{\pi} \coloneqq (\overline{\pi}_2,\dots,\overline{\pi}_n)$: it is a regular system of parameters in the regular local ring $R/(\pi_1)$. Then s_π is defined by $s_\pi \coloneqq s_{\overline{\pi}} \circ s_{(\pi_1)}$, where π_1 is viewed as an element of the localization $R_{(\pi_1)}$.

The ring homomorphism s_π depends on the choice of the ordered set π. Using the isomorphism $H^2(F,\mathbb{Z}/2\mathbb{Z}) \simeq \mathrm{Br}(F)[2]$ coming from Kummer theory, we obtain a specialization map

$$s_\pi\colon \mathrm{Br}(K)[2] \to \mathrm{Br}(K_0)[2].$$

Let X be an F-variety (that is, a separated integral F-scheme of finite type) and $P \in X$ be a regular F-point. For all regular systems of parameters $\pi = (\pi_1,\dots,\pi_n)$ in the regular local ring $R = O_{X,P}$ the previous discussion yields specialization maps

$$s_{P,\pi}\colon H^*(F(X),\mathbb{Z}/2\mathbb{Z}) \to H^*(F,\mathbb{Z}/2\mathbb{Z}), \qquad s_{P,\pi}\colon \mathrm{Br}(F(X))[2] \to \mathrm{Br}(F)[2].$$

If $f \in O_{X,P}^\times$ (that is, f is regular and nonzero at P) then it follows from the definition that $s_{P,\pi}(f) = (f(P))$. In particular, if $f \in F^\times$ is constant then $s_{P,\pi}(f) = (f)$.

Lemma 3.1. *Let $n \geq 1$ be an integer, X be an n-dimensional F-variety, $P \in X$ be a regular F-point, and $\pi \coloneqq (\pi_1,\dots,\pi_n)$ be a regular system of parameters in $O_{X,P}$. Let F' be a finite separable field extension of F, let $X' \coloneqq X \times_F F'$, let P' be the only F'-point of X' lying over P, and consider the system of parameters $\pi' \coloneqq (\pi_1 \otimes 1,\dots,\pi_n\otimes 1)$ in the regular local ring $O_{X',P'} = O_{X,P}\otimes_F F'$. Then the following squares commute:*

$$\begin{tikzcd}
H^*(F(X),\mathbb{Z}/2\mathbb{Z}) \arrow[r, "s_{P,\pi}"] \arrow[d, "(-)_{F'(X')}"] & H^*(F,\mathbb{Z}/2\mathbb{Z}) \arrow[d, "(-)_{F'}"] & H^*(F'(X'),\mathbb{Z}/2\mathbb{Z}) \arrow[r, "s_{P',\pi'}"] \arrow[d, "N_{F'(X')/F(X)}"] & H^*(F',\mathbb{Z}/2\mathbb{Z}) \arrow[d, "N_{F'/F}"] \\
H^*(F'(X'),\mathbb{Z}/2\mathbb{Z}) \arrow[r, "s_{P',\pi'}"] & H^*(F',\mathbb{Z}/2\mathbb{Z}) & H^*(F(X),\mathbb{Z}/2\mathbb{Z}) \arrow[r, "s_{P,\pi}"] & H^*(F,\mathbb{Z}/2\mathbb{Z}).
\end{tikzcd}$$

Lemma 3.1 admits an obvious generalization to the case when F' is an étale F-algebra.

Proof. One proves the result by induction on $n \geq 1$; see [MS23a, Lemma 2.9]. □

3.2. **A calculation.** Let F be a field of characteristic different from 2, and let c, x_1, x_2, y_1, y_2 and u be variables over F. Consider the polynomials

$$\begin{aligned} d &= u^2 - c, \\ w &= x_1y_2 + x_2y_1, \\ h &= x_1y_1 + ux_1y_2 + ux_2y_1 + cx_2y_2. \end{aligned}$$

Note that these polynomials are symmetric with respect to the change of variables $x_i \leftrightarrow y_i$.

Proposition 3.2. *Let F be a field of characteristic different from 2, let c, x_1, x_2, y_1, y_2 and u be variables over F, and let $L := F(c, x_1, x_2, y_1, y_2, u)$. Then we have the following equality in* $\mathrm{Br}(L)[2]$*:*

$$\begin{aligned} \Big((x_1^2 - cx_2^2)(y_1^2 - cy_2^2), 2wh\Big) = &\Big(x_1^2 - cx_2^2, 2x_2(x_1 + ux_2)\Big) \\ &+ \Big(y_1^2 - cy_2^2, 2y_2(y_1 + uy_2)\Big) \\ &+ \Big(d, (x_1 + ux_2)(y_1 + uy_2)h\Big). \end{aligned}$$

Proof. We have

$$x_2h + (x_1^2 - cx_2^2)y_2 = w(x_1 + ux_2). \tag{3.1}$$

Indeed,

$$\begin{aligned} x_2h + (x_1^2 - cx_2^2)y_2 &= x_2(x_1y_1 + ux_1y_2 + ux_2y_1 + cx_2y_2) + (x_1^2 - cx_2^2)y_2 \\ &= (x_1y_2 + x_2y_1)(x_1 + ux_2) \\ &= w(x_1 + ux_2). \end{aligned}$$

Symmetrically, we get the equality

$$y_2h + (y_1^2 - cy_2^2)x_2 = w(y_1 + uy_2). \tag{3.2}$$

We deduce from (3.1) and (3.2) that

$$(x_1^2 - cx_2^2)(y_1^2 - cy_2^2)x_2y_2 = (w(x_1 + ux_2) - x_2h)(w(y_1 + uy_2) - y_2h). \tag{3.3}$$

Note that

$$h = (x_1 + ux_2)(y_1 + uy_2) - dx_2y_2. \tag{3.4}$$

Combining (3.1), (3.2) and (3.4), we get the equality

$$(x_2h + (x_1^2 - cx_2^2)y_2) \cdot (y_2h + (y_1^2 - cy_2^2)x_2) = w^2(h + dx_2y_2). \tag{3.5}$$

We have

$$x_1^2 - cx_2^2 = (x_1 + ux_2)(x_1 - ux_2) + dx_2^2, \tag{3.6}$$

and symmetrically

$$y_1^2 - cy_2^2 = (y_1 + uy_2)(y_1 - uy_2) + dy_2^2. \tag{3.7}$$

We prove that the residues of both sides of the equality with respect to every irreducible polynomial $p \in F[c, x_1, x_2, y_1, y_2, u]$ are equal.

1. The cases $p = x_1 + ux_2$ (resp. $p = y_1 + uy_2$) follow from (3.6) (resp. (3.7)).
2. The cases $p = x_1^2 - cx_2^2$ (resp. $p = y_1^2 - cy_2^2$) follows from (3.1) (resp. (3.2)).
3. The case $p = h$ follows from (3.5).
4. The case $p = w$ follows from (3.3).

(5) The case $p = d$ follows from (3.4).
(6) The cases $p = x_2$ (resp. $p = y_2$) are obvious.
(7) The case when p is any other polynomial is obvious.

This shows that the two sides differ by a constant Brauer class, that is, a class in the image of the map $\mathrm{Br}(F) \to \mathrm{Br}(L)$. Since the equality holds after specializing at $c = 0$, the proof is complete. □

Lemma 3.3 (Trace Lemma). *Let $\rho \in F_a^\times$ and $\mu \in F_b^\times$ be such that $N_a(\rho) = N_b(\mu)$. Set $g := \mathrm{Tr}_a(\rho) + \mathrm{Tr}_b(\mu)$, and suppose that $d \neq 0$. Then $(\mu, a) = (g, a)$ in $\mathrm{Br}(F_b)$.*

Proof. We have

$$\begin{aligned} N_a(\rho + \mu) &= (\rho + \mu)(\sigma_a(\rho) + \mu) \\ &= \rho\sigma_a(\rho) + \rho\mu + \mu\sigma_a(\rho) + \mu^2 \\ &= \mu\sigma_b(\mu) + \rho\mu + \mu\sigma_a(\rho) + \mu^2 \\ &= \mu(\mathrm{Tr}_a(\rho) + \mathrm{Tr}_b(\mu)) \\ &= \mu g. \end{aligned}$$

It follows that $(\mu g, a) = (N_a(\rho + \mu), a) = 0$ in $\mathrm{Br}(F_b)$, that is, $(\mu, a) = (g, a)$ in $\mathrm{Br}(F_b)$. □

3.3. **Proof of Proposition 2.5.**

Proof of Proposition 2.5. Because $(\alpha, c) = 0$, there exist $\alpha_1, \alpha_2 \in F_a^\times$ such that

$$\alpha = \alpha_1^2 - c\alpha_2^2.$$

Claim 3.4. We may suppose α_1, α_2 linearly independent over F.

Proof of Claim 3.4. Suppose that α_1 and α_2 are linearly dependent over F, so that there exists $t \in F$ such that either $\alpha_1 = t\alpha_2$ or $\alpha_2 = t\alpha_1$. In the first case $\alpha = (t^2 - c)\alpha_2^2$, and in the second case $\alpha = (1 - ct^2)\alpha_1^2$. Thus, there exist $i \in \{1, 2\}$ and $u \in F^\times$ such that $\alpha = u\alpha_i^2$. Note that $u \in F^\times$ and $\alpha_i \in F_a^\times$ because $\alpha \in F_a^\times$. Letting $x = u$ and $\nu = 1$, we have $(\alpha x, \delta) = (u^2, \delta) = 0$ in $\mathrm{Br}(F_{a,d})$ and $(\alpha x, \nu) = (ux, \nu) = 0$ in $\mathrm{Br}(F_a)$, which proves Proposition 2.5 in this case. □

From now on, we assume that α_1 and α_2 are linearly independent over F. Let $K := F(\mathbb{A}^2) = F(x_1, x_2)$, and define

$$f := x_1^2 - cx_2^2 \in K^\times.$$

Let

$$h_1 := \alpha_1 x_1 + c\alpha_2 x_2 \in K_a^\times, \qquad h_2 := \alpha_1 x_2 + \alpha_2 x_1 \in K_a^\times.$$

Let $u_1, u_2 \in F$ be such that

$$\delta = u_1 + u_2\sqrt{d},$$

so that $N_d(\delta) = u_1^2 - du_2^2 = c$. We define the following elements of $F_a^\times$:

$$\beta_1 := \alpha_1 + u_1\alpha_2, \qquad \beta_2 := u_1\alpha_1 + c\alpha_2, \qquad \theta := 2\alpha_2\beta_1.$$

Finally, we define

$$g := 2hh_2 \in K_a^\times, \qquad t := x_1 + u_1x_2 \in K^\times, \qquad s := 2x_2t = 2(x_1x_2 + u_1x_2^2) \in K^\times.$$

Lemma 3.5. *We have $(\alpha, \theta) = (\alpha, \delta)$ and $(\alpha f, g) = (\alpha f, \delta)$ in $\mathrm{Br}(K_{a,d})[2]$.*

Proof. Set $\rho := (\alpha_1 + \sqrt{\alpha})\alpha_2{}^{-1}$. We have $N_\alpha(\rho) = c = N_d(\delta)$. The equality

$$\mathrm{Tr}_\alpha(\rho) + \mathrm{Tr}_d(\delta) = 2(\alpha_1\alpha_2^{-1} + u_1) = 2\alpha_2^{-1}\beta_1,$$

and Lemma 3.3 imply that $(\alpha, \theta) = (\alpha, 2\alpha_2\beta_1) = (\alpha, \delta)$ over $F_{a,d}$. The proof that $(\alpha f, g) = (\alpha f, \delta)$ over $K_{a,d}$ is similar. □

Specialization of the equality of Proposition 3.2 at $y_1 = \alpha_1$, $y_2 = \alpha_2$ and $u = u_1$ yields

$$(f\alpha, 2h_2h) = (f, 2x_2t) + (\alpha, 2\alpha_2\beta_1) + (d, t\beta_1h) \quad \text{in } \mathrm{Br}(K_a)[2],$$

or equivalently

$$(\alpha f, g) + (f, s) + (d, \beta_1ht) + (\alpha, \theta) = 0 \quad \text{in } \mathrm{Br}(K_a)[2]. \tag{3.8}$$

So far, we have not yet used the fact that (α, δ) comes from $\mathrm{Br}(F)[2]$. Let $A \in \mathrm{Br}(F)[2]$ be such that $(\alpha, \delta) = A_{F_{a,d}}$ over $F_{a,d}$. Then $(\alpha, \theta) - A_{F_a}$ in $\mathrm{Br}(F_a)$ vanishes over $F_{a,d}$, and hence $(\alpha, \theta) - A_{F_a} = (d, \epsilon)$ for some $\epsilon \in F_a^\times$. Applying $N_{K_a/K}$ to (3.8), we get

$$N_{K_a/K}(\alpha f, g) = (d, N_{K_a/K}(h\eta)),$$

where $\eta \coloneqq \epsilon\beta_1 \in F_a^\times$. Let $P = (P_1, P_2) \in \mathbb{A}^2_F(F)$ be an F-point such that h is regular and invertible at P, let $\pi \coloneqq (x_1 - P_1, x_2 - P_2)$ be a system of parameters at P, and define the specializations $x \coloneqq s_\pi(f) \in F^\times$ and $\nu \coloneqq s_\pi(g)$. We specialize the above equation at P via π to obtain

$$N_{F_a/F}(\alpha x, \nu) = (d, N_{F_a/F}(h(P)\eta)).$$

We wish to find P so that the right-hand side is zero. This would be case if we could choose P such that h is regular at P and $h(P) = \eta$. We have

$$\begin{aligned} h(P) = \eta &\iff u_1(\alpha_1P_2 + \alpha_2P_1) + (\alpha_1P_1 + c\alpha_2P_2) = \eta \\ &\iff (\alpha_1 + u_1\alpha_2)P_1 + (u_1\alpha_1 + c\alpha_2)P_2 = \eta. \end{aligned}$$

Recall that α_1 and α_2 were chosen to be linearly independent over F. Thus it suffices to check that

$$\det \begin{bmatrix} 1 & u_1 \\ u_1 & c \end{bmatrix} = c - u_1^2$$

is not zero. This is true because c is not square in F. Thus we may find P such that $h(P) = \eta$: in fact, we showed that P exists and is unique. □

Propositions 2.4, 2.5 and 2.6 have thus been proved, and hence the proof of Theorem 1.9 is complete.

4. Lecture 4. Formal Hilbert 90 and non-formality of Galois cohomology

4.1. **Formal Hilbert 90.** In this final lecture, we wish to examine the following vague question.

Question 4.1. *Is the Massey Vanishing Conjecture a consequence of Hilbert's Theorem 90 alone?*

Here is one way to make this question precise. Let p be a prime number, let $\mathbb{Z}_p$ be the ring of p-adic integers, and let Γ be a profinite group, and let $\theta\colon \Gamma \to \mathbb{Z}_p^\times$ be a continuous group homomorphism. We call θ a *p-orientation* of Γ and the pair (Γ, θ) a *p-oriented profinite group.*

We write $\mathbb{Z}_p(1)$ for the topological Γ-module with underlying topological group $\mathbb{Z}_p$ on which Γ acts via θ, that is, $g \cdot v \coloneqq \theta(g)v$ for every $g \in \Gamma$ and every $v \in \mathbb{Z}_p$. For all $n \geq 0$, we set $\mathbb{Z}/p^n\mathbb{Z}(1) \coloneqq \mathbb{Z}_p(1)/p^n\mathbb{Z}_p(1)$.

Let (Γ, θ) be a p-oriented profinite group. We say that (Γ, θ) *satisfies formal Hilbert 90* if for every open subgroup $H \subset \Gamma$ and all $n \geq 1$ the reduction map $H^1(H, \mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(H, \mathbb{Z}/p\mathbb{Z}(1))$ is surjective.

Example 4.2. Let F be a field and write Γ_F for the absolute Galois group of F. We define the *canonical p-orientation* θ_F on Γ_F as follows. If $\mathrm{char}(F) \neq p$, we define θ_F as the continuous homomorphism $\theta_F\colon \Gamma_F \to \mathbb{Z}_p^\times$ given by $g(\zeta) = \zeta^{\theta_F(g)}$ for every root of unity ζ of p-power order in F_s. If $\mathrm{char}(F) = p$, we let θ_F be the trivial homomorphism. The pair (Γ_F, θ_F) is a p-oriented profinite group and, by Hilbert's Theorem 90, it satisfies formal Hilbert 90.

We may now formulate Question 4.1 in a more precise way.

Question 4.3. *Let p be a prime number, let (Γ, θ) be a p-oriented profinite group which satisfies formal Hilbert 90, let $n \geq 3$, and let $\chi_1, \dots, \chi_n \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$. If $\langle \chi_1, \dots, \chi_n \rangle$ is defined, does it vanish?*

In [MS25], we proved that Question 4.3 has affirmative answer when $n = 3$, as well as when $(n, p) = (4, 2)$ and $\chi_1 = \chi_4$ (the degenerate case).

Theorem 4.4 (Merkurjev–Scavia)**.** *Let p be a prime number, let (Γ, θ) be a p-oriented profinite group satisfying formal Hilbert 90 and let $\chi_1, \chi_2, \chi_3 \in H^1(\Gamma, \mathbb{Z}/p\mathbb{Z})$. The following are equivalent:*

(1) $\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = 0$ in $H^2(\Gamma, \mathbb{Z}/p\mathbb{Z})$;
(2) the mod p Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle$ is defined;
(3) the mod p Massey product $\langle \chi_1, \chi_2, \chi_3 \rangle$ vanishes.

Theorem 4.5 (Merkurjev–Scavia)**.** *Let (Γ, θ) be a 2-oriented profinite group satisfying formal Hilbert 90 and let $\chi_1, \chi_2, \chi_3 \in H^1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. If the mod 2 Massey product $\langle \chi_1, \chi_2, \chi_3, \chi_1 \rangle$ is defined, then it vanishes.*

Remark 4.6. In particular, Theorem 4.5 gives an alternative proof of Theorem 1.9, in the degenerate case $\chi_1 = \chi_4$, which does not rely on the theory of Albert forms but only uses Hilbert's Theorem 90.

Beyond the degenerate case $\chi_1 = \chi_4$, we do not know whether Theorem 1.9 can be extended to 2-oriented profinite groups satisfying formal Hilbert 90. This is because we do not know how to generalize Proposition 2.5 to this setting.

Let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ be the localization of $\mathbb{Z}$ at the prime ideal (p). The group $\mathbb{Z}_p^\times$ acts on the abelian group $\mathbb{Q}/\mathbb{Z}_{(p)}$ by multiplication. We let S be the Γ-module whose underlying abelian group is $\mathbb{Q}/\mathbb{Z}_{(p)}$ and on which Γ acts via θ. For all $n \geq 1$, we have an isomorphism of Γ-modules $\mathbb{Z}/p^n\mathbb{Z}(1) \to S[p^n]$ given by $a + p^n\mathbb{Z} \mapsto a/p^n + \mathbb{Z}_{(p)}$. Therefore, S is the colimit of the $\mathbb{Z}/p^n\mathbb{Z}(1)$ for $n \geq 1$.

The key tool for the proof of Theorems 4.4 and 4.5 is contained in the following definition.

Definition 4.7. Let (Γ, θ) be a p-oriented profinite group. A *Hilbert 90 module* for (Γ, θ) is a discrete Γ-module M such that

(i) $pM = M$,
(ii) the p-primary torsion subgroup of M is isomorphic to S as a Γ-module, and
(iii) $H^1(H, M) = 0$ for every open subgroup $H \subset \Gamma$.

Example 4.8. Let p be a prime number, let F be a field of characteristic different from p, let Γ_F be the absolute Galois group of F, and let θ be the canonical orientation on Γ_F; see Example 4.2. It follows from Hilbert's Theorem 90 that $F_s^\times$ is a Hilbert 90 module for (Γ_F, θ_F).

It turns out that every p-oriented profinite group satisfying formal Hilbert 90 admits a Hilbert 90 module.

Theorem 4.9 (Merkurjev–Scavia). *Let (Γ, θ) be a p-oriented profinite group. Then (Γ, θ) satisfies formal Hilbert 90 if and only if it admits a Hilbert 90 module.*

The most difficult part of the proof of Theorem 4.9 is to construct a Hilbert 90 module for a pair (Γ, θ) which satisfies formal Hilbert 90. When $(\Gamma, \theta) = (\Gamma_F, \theta_F)$, it is not clear how the Hilbert 90 module constructed in Theorem 4.9 is related to the one of Example 4.8.

With Theorem 4.9 at our disposal, we may try to adapt the proofs of the Massey Vanishing Conjecture in the $n = 3$ case or in the degenerate $(n, p) = (4, 2)$ case. In the first case, one must replace the arguments involving central simple algebras split by a $(\mathbb{Z}/p\mathbb{Z})^2$-extension by cocycle arguments. In the second case, the key point is to prove Proposition 2.6 without quadratic form theory. The point is that Proposition 2.6 may also be proved from suitable exact sequences of F-tori, and such a proof may be mimicked in this more general setting. Indeed, if T is an F-torus with cocharacter lattice T_*, then $T(F_s) = T_* \otimes F_s^\times$ may be expressed using only the Γ_F-lattice T_* and the Hilbert 90 module $F_s^\times$.

4.2. **Non-formality of Galois cohomology: Positselski's question.** Let (A, ∂) be a differential graded ring, i.e, $A = \oplus_{i \geq 0} A^i$ is a non-negatively graded abelian group with an associative multiplication which respects the grading, and $\partial\colon A \to A$ is a degree 1 homomorphism of graded groups such that $\partial \circ \partial = 0$ and

$$\partial(ab) = \partial(a)b + (-1)^i a\partial(b) \quad \text{for all } i \geq 0,\ a \in A^i \text{ and } b \in A.$$

We say that A is *formal* if it is quasi-isomorphic as a differential graded ring to $H^*(A)$ with the zero differential, that is, if there exist a differential graded ring B and a diagram

$$A \longleftarrow B \longrightarrow H^*(A),$$

where both maps are quasi-isomorphisms. Loosing speaking, A is formal if no essential information about A is lost when passing to $H^*(A)$.

Hopkins–Wickelgren [HW15] asked whether $C^\bullet(\Gamma_F, \mathbb{Z}/p\mathbb{Z})$ is formal for every field F and every prime p. The authors of [HW15] were unaware of earlier work of Positselski, who had already showed in [Pos10, Section 9.11] that $C^\bullet(\Gamma_F, \mathbb{Z}/p\mathbb{Z})$ is not formal for some finite extensions F of $\mathbb{Q}_\ell$ and $\mathbb{F}_\ell((z))$, where $\ell \neq p$. Positselski later wrote a detailed exposition of his counterexamples in [Pos17].

For Positselski's method to work, it seemed important that F did not contain all the roots of unity of p-power order. This motivated the following question; see [Pos17, p. 226].

Question 4.10 (Positselski)**.** *Does there exist a field F containing all roots of unity of p-power order such that $C^{\bullet}(\Gamma_F, \mathbb{Z}/p\mathbb{Z})$ is not formal?*

Using Massey products, we were able to show in [MS23b] that Question 4.10 has negative answer in general.

Theorem 4.11 (Merkurjev–Scavia)**.** *Let p be a prime number and let F be a field of characteristic different from p. There exists a field L containing F such that the differential graded ring $C^{\bullet}(\Gamma_L, \mathbb{Z}/p\mathbb{Z})$ is not formal.*

We devote the remainder of this lecture to the proof of Theorem 4.11.

4.3. **Massey products and formality.** In order to explain the relation between Question 4.10 and Massey products, it is necessary to discuss Massey products for an arbitrary differential graded ring A, beyond the case of mod p continuous cochains modulo p that has been considered so far in these notes. Let $n \geq 2$ be an integer, and let $a_1, \dots, a_n \in H^1(A)$. By definition, a *defining system* for the Massey product $\langle a_1, \dots, a_n \rangle$ is a collection M of elements of $a_{ij} \in A^1$, where $1 \leq i < j \leq n+1$, $(i, j) \neq (1, n+1)$, such that

(1) $\partial(a_{i,i+1}) = 0$ and $a_{i,i+1}$ represents a_i in $H^1(A)$, and
(2) $\partial(a_{ij}) = -\sum_{l=i+1}^{j-1} a_{il}a_{lj}$ for all $i < j-1$.

It follows from (2) that $-\sum_{l=2}^{n} a_{1l}a_{l,n+1}$ is a 2-cocycle: we write $\langle a_1, \dots, a_n \rangle_M$ for its cohomology class in $H^2(A)$, called the *value* of $\langle a_1, \dots, a_n \rangle$ corresponding to M.

Definition 4.12. The *Massey product* of $a_1, \dots, a_n$ is the subset $\langle a_1, \dots, a_n \rangle$ of $H^2(A)$ consisting of the values $\langle a_1, \dots, a_n \rangle_M$ of all defining systems M. We say that the Massey product $\langle a_1, \dots, a_n \rangle$ is *defined* if it is non-empty, and that it *vanishes* if $0 \in \langle a_1, \dots, a_n \rangle$.

By a theorem of Dwyer [Dwy75], this definition reduces to Definition 1.1 when A is the cochain DGA of a profinite group.

Lemma 4.13. *Let (A, ∂) be a differential graded ring, let $n \geq 3$ be an integer, and let $a_1, \dots, a_n$ be elements of $H^1(A)$ satisfying $a_i \cup a_{i+1} = 0$ for all $1 \leq i \leq n-1$. If A is formal, then $\langle a_1, \dots, a_n \rangle$ vanishes.*

Proof. See [PQ25, Theorem 3.8]. It is not difficult to give a direct proof in the case $n = 4$, which is the only case that we will need. □

Recall that the Massey Vanishing Conjecture asks whether the first implication of Proposition 1.3 can be reversed for absolute Galois groups. Lemma 4.13 tells us that, for a formal DGA A, both implications of Proposition 1.3 can be reversed. It is natural to wonder whether both implications can be reversed for absolute Galois groups; see [MT17b, Question 4.2] and [PS18, Definition 1.3]. This leads us to the following question of Mináč and Tân.

Question 4.14 (Strong Massey Vanishing (Mináč–Tân))**.** *Let F be a field, let $n \geq 3$ be an integer, let p be a prime number, and let $\chi_1, \dots, \chi_n \in H^1(F, \mathbb{Z}/p\mathbb{Z})$ be such that $\chi_i \cup \chi_{i+1} = 0$ for all $i = 1, \dots, n$. Does $\langle \chi_1, \dots, \chi_n \rangle$ vanish?*

It is clear that if Strong Massey Vanishing is true for F, then the Massey Vanishing Conjecture holds for F. Moreover, by Lemma 4.13, if the Strong Massey Vanishing Conjecture fails, for some $n \geq 3$ and some prime p, then $C^{\bullet}(\Gamma_F, \mathbb{Z}/p\mathbb{Z})$

is not formal. Therefore, in order to prove Theorem 4.11, it suffices to exhibit a field L for which Strong Massey Vanishing fails.

Before our work, the only known example of a field for which Strong Massey Vanishing fails was $F = \mathbb{Q}$, as shown by Harpaz and Wittenberg [GMT18, Example A.15].

Example 4.15 (Harpaz–Wittenberg)**.** Strong Massey Vanishing fails for $F = \mathbb{Q}$, $n = 4$, and $p = 2$. More precisely, if we let $b = 2$, $c = 17$ and $a = d = bc = 34$, then $(a, b) = (b, c) = (c, d) = 0$ in $\mathrm{Br}(\mathbb{Q})$ but $\langle a, b, c, d\rangle$ is not defined over $\mathbb{Q}$.

As a first attempt towards the proof of Theorem 4.11, it is natural to try to generalize the Harpaz–Wittenberg example to arbitrary fields. This is our [MS22, Theorem 1.4].

Theorem 4.16 (Merkurjev–Scavia)**.** *Let $p = 2$, let F be a field of characteristic different from 2, and let $b, c \in F^\times$. The following are equivalent:*

1. *the Massey product $\langle bc, b, c, bc\rangle$ is defined,*
2. *the Massey product $\langle bc, b, c, bc\rangle$ vanishes,*
3. *$(b, c) = 0$ in $\mathrm{Br}(F)$ and $-1 \in N_{b,c}(F^\times_{b,c})$.*

Theorem 4.16 does not imply Theorem 4.11, that is, it does not suffice to give a negative answer to Question 4.10. Indeed, if F contains a primitive 8-th root of unity ζ_8, then $-1 = N_{b,c}(\zeta_8) \in N_{b,c}(F^\times_{b,c})$, and hence if $(b, c) = 0$ then $\langle bc, b, c, bc\rangle$ vanishes by Theorem 4.16.

Theorem 4.11 is a consequence of the next more precise result, which we proved in [MS23b, Theorem 1.3].

Theorem 4.17 (Merkurjev–Scavia)**.** *Let p be a prime number, let F be a field of characteristic different from p. There exist a field L containing F and $\chi_1, \chi_2, \chi_3, \chi_4 \in H^1(L, \mathbb{Z}/p\mathbb{Z})$ such that $\chi_1 \cup \chi_2 = \chi_2 \cup \chi_3 = \chi_3 \cup \chi_4 = 0$ in $H^2(L, \mathbb{Z}/p\mathbb{Z})$ but $\langle \chi_1, \chi_2, \chi_3, \chi_4\rangle$ is not defined. Thus the Strong Massey Vanishing property relative to $n = 4$ and the prime p fails for L, and $C^\cdot(\Gamma_L, \mathbb{Z}/p\mathbb{Z})$ is not formal.*

We construct L and the χ_i. Replacing F by a finite extension if necessary, we may suppose that F contains a primitive p-th root of unity ζ. Let $E := F(x, y)$, where x and y are independent variables over F, let X be the Severi-Brauer variety of the degree-p cyclic algebra (x, y) over E, and let $L := E(X)$. Consider the following elements of $E^\times$:

$$a := 1 - x, \quad b := x, \quad c := y, \quad d := 1 - y.$$

We have $(a, b) = (c, d) = 0$ in $\mathrm{Br}(E)$ by the Steinberg relations [Ser79, Chapter XIV, Proposition 4(iv)], and hence $(a, b) = (b, c) = 0$ in $\mathrm{Br}(L)$. Moreover, $(b, c) \neq 0$ in $\mathrm{Br}(E)$ because the residue of (b, c) along $x = 0$ is non-zero, while $(b, c) = 0$ in $\mathrm{Br}(L)$ by [GS17, Theorem 5.4.1]. Thus $(a, b) = (b, c) = (c, d) = 0$ in $\mathrm{Br}(L)$. In order to prove Theorem 4.17, it suffices to prove that $\langle a, b, c, d\rangle$ is not defined. We summarize the main steps of the proof.

The first step is to find an equivalent condition for the property "$\langle a, b, c, d\rangle$ is defined" in the spirit of Proposition 2.4.

Proposition 4.18. *Let p be a prime, let F be a field of characteristic different from p and containing a primitive p-th root of unity ζ, and let $a, b, c, d \in F^\times$. The mod*

p Massey product $\langle a,b,c,d\rangle$ is defined if and only if there exist $u\in F_{a,c}^{\times}$, $v\in F_{b,d}^{\times}$ and $w_0\in F_{b,c}^{\times}$ such that

$$N_a(u)\cdot N_d(v)=w_0^p, \qquad (\sigma_b-1)(\sigma_c-1)w_0=\zeta.$$

Proof sketch. We refer to [MS23b, Proposition 3.7] for the complete proof of Proposition 4.18. The idea is the following. The Massey product $\langle a,b,c,d\rangle$ is defined if and only if there exists a Galois $\overline{U}_5$-algebra L/F with induced $(\mathbb{Z}/p\mathbb{Z})^4$-algebra $F_{a,b,c,d}/F$. Contemplating the following picture of $\overline{U}_5$

$$\begin{bmatrix} 1 & * & * & * & \square \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.1}$$

we see that this is equivalent to the existence of a U_4-algebra inducing $F_{a,b,c}/F$ (top-left 4×4 square), a U_4-algebra inducing $F_{b,c,d}/F$ (bottom-right 4×4 square), and an isomorphism of the induced U_3-algebras (central 3×3 square). The strategy is to parametrize all possibilities for the U_4-algebras, and to impose the condition that they agree on the common U_3 square. Loosely speaking, u corresponds to the upper U_4-square, v to the bottom U_4-square, and w_0 to the fact that the two U_4-squares agree on the common U_3-square. ☐

Once Proposition 4.18 is established, elementary calculations yield the following.

Corollary 4.19. *Let p be a prime, let F be a field of characteristic different from p and containing a primitive p-th root of unity ζ, let $a,b,c,d\in F^{\times}$, and suppose that $\langle a,b,c,d\rangle$ is defined over F. For every $w\in F_{b,c}^{\times}$ such that $(\sigma_b-1)(\sigma_c-1)w=\zeta$, there exist $u\in F_{a,c}^{\times}$ and $v\in F_{b,d}^{\times}$ such that $N_a(u)N_d(v)=w^p$.*

Proof. See [MS23b, Corollary 4.5]. ☐

We can rephrase Corollary 4.19 as follows. Let $\mathcal{T}$ be the kernel of the homomorphism of F-tori

$$R_{a,c}(\mathbb{G}_m)\times R_{b,d}(\mathbb{G}_m)\to R_{b,c}(\mathbb{G}_m), \qquad (u,v)\mapsto N_a(u)N_d(v)=1.$$

Here $R_{a,c}(\mathbb{G}_m)$ (resp. $R_{b,d}(\mathbb{G}_m)$) denotes the Weil restriction of $\mathbb{G}_m$ from $F_{a,c}$ (resp. $F_{b,d}$) to F: it is a quasi-trivial F-torus of rank 4. One shows that $\mathcal{T}$ is an F-torus, that is, it is connected; see [MS23b, Lemma 4.3]. Given $w\in F_{b,c}^{\times}$ such that $(\sigma_b-1)(\sigma_c-1)w=\zeta$, the Massey product $\langle a,b,c,d\rangle$ is defined if and only if the T-torsor $E_w\subset R_{a,c}(\mathbb{G}_m)\times R_{b,d}(\mathbb{G}_m)$ given by $N_a(u)N_d(v)=w^p$ is split.

More generally, suppose that T is a torus over a field F, let K be a Galois field extension of F such that T_K is split, and let $G=\mathrm{Gal}(K/F)$. We have an exact sequence of G-modules

$$1\to T(K)\to T(K(X))\xrightarrow{\mathrm{div}}\mathrm{Div}(X_K)\otimes T_*\xrightarrow{\mathrm{deg}}T_*\to 0, \tag{4.2}$$

where T_* denotes the cocharacter lattice of T. We consider the subgroup of unramified torsors

$$H^1(G,T(K(X)))_{\mathrm{nr}}\coloneqq \mathrm{Ker}[H^1(G,T(K(X)))\xrightarrow{\mathrm{div}}H^1(G,\mathrm{Div}(X_K\otimes T_*))],$$

and the homomorphism

$$\theta\colon H^1(G, T(K(X)))_{\mathrm{nr}} \to \mathrm{Coker}[(\mathrm{Div}(X_K) \otimes T_*)^G \xrightarrow{\deg} (T_*)^G],$$

induced by (4.2). It turns out that it is possible to compute θ explicitly in terms of any short exact sequence

$$1 \to T \to P \to S \to 1$$

where P is a quasi-trivial torus; see [MS23b, Appendix B] for details.

In our setup $F = E$, $\mathcal{T} = T$, $K = E_{a,b,c,d}$, and $P = R_{a,c}(\mathbb{G}_m) \times R_{b,d}(\mathbb{G}_m)$. By lattice computations, we show in [MS23b, §5] that the T_L-torsor E_w is unramified and that $\theta([E_w]) \neq 0$: more precisely, in our example the codomain of θ is $\mathbb{Z}/p\mathbb{Z}$ and $\theta([E_w])$ is a generator. Therefore E_w is non-trivial, and hence $\langle a, b, c, d\rangle$ is not defined, completing the proof of Theorem 4.11.

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